THE LOGIC LINGUISTS NEED TO KNOW

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Introduction

Lewis Carroll's logical problems have long provided a rich source of examples for teachers of logic, and there have been many more of these problems to choose from since W. W. Bartley III rediscovered and published more of Carroll's logical work (Bartley, 1977).

Bartley also provides an interesting editor's introduction to this publication. In it, he notes a point which is often overlooked; namely, that the "Boolean" or algebraic logicians of the 19th century were trying to answer rather different problems from the ones which are tackled by modern mathematical logicians. Boole himself, it seems, defined the central problem of algebraic logic as follows (see Bartley, 1977, p. 22):

"Given certain logical premises or conditions, to determine the description of any class of objects under those conditions."

Contemporary logicians, on the other hand, are generally concerned with problems in the foundations of mathematics. These involve the construction of proofs and axiomatization, with the attendant split between logical syntax and semantics, and the resulting problems of decision procedures and their limitations. The methods used in these investigations are rather different from those required for the Boolean problem.

Bartley explains the general lack of awareness of this distinction today by the Kuhnian suggestion that 19th century algebraic logic, although certainly "revolutionary" when compared to the current Aristotelian variety, never had time to become a "paradigm" before the advent of mathematical logic in the early 1900's. (See Kuhn, 1970, for these terms.)

I think it is clear that linguists investigating logical form are far more concerned with a Boolean problem that with a mathematical one. For example, consider again the classic illustration of Montague grammar, "John seeks a unicorn." A semantic theory should indicate, among other things, that this sentence may be true even if unicorns do not exist, and, if unicorns are necessarily animals, that it implies "John seeks an animal." Montague tackles the problem by translating the English sentences into an interpreted intensional logic, and showing that the translation has the necessary implications. This is a Boolean method: the axiomatizations and syntactic proof procedures of mathematical logic are not necessary for its success. In fact, Montague's linguistic papers do not offer any proof procedures at all, but rely on intuitive considerations. As for axiomatization, Montague writes in his 'Pragmatics and Intensional Logic', "The problem, however, of axiomatizing predicative intensional logic remains open." (See Thomason, 1974, p. 144.) This lack did not hinder a solution to the linguistic problem.

Despite all this, works on logic for linguists are largely based on those intended for mathematicians. McCawley (1981), for example, has separate chapters on syntax, preceding those on semantics, for propositional and predicate logic. The syntax chapters contain cumbersome methods of proof based on natural deduction, which despite its name, needs a good deal of skill and memorization to apply freely. Furthermore, the natural deductive methods are not used in the chapter on Montague grammar. It is noteworthy too that the syntax chapters of the book contain a great deal of comment on intended interpretations of the theorems being proved. Allwood, Andersson, and Dahl (1977) does not contain any complete method of proof even for lower predicate logic, although in general the authors do stress semantics rather than syntax.

In what follows I shall present what I hope is a solution to the Boolean problem for linguists using logic. It is based on an adaptation of a proof procedure for the lower predicate calculus used by Quine in his (1974). Since there is no linguistic reason to restrict myself to an axiomatic, syntactic justification of this proof procedure, I will give only a simple semantic one. In a future paper I shall show how this method may easily be extended to deal with problems in modal and intensional logic, which I believe greatly simplifies the conceptual difficulties involved in mastering Montague grammar.

The Method

Let us begin with the simple problem of showing that a conclusion $(x)\ (Fx \supset Hx)$ follows from the two premises $(x)\ (Gx \supset Hx)$ and $(x)\ (Fx \supset Gx)$: a basic Aristotelian syllogism. In other terms, this amounts to showing that the sentence

$$(x)\ (Gx \supset Hx) \cdot (x)\ (Fx \supset Gx) \cdot \supset (x)\ (Fx \supset Hx)$$

is logically true, or true under any interpretation.

One way of showing that something is logically true is to demonstrate that its negation would be inconsistent, or false under any interpretation. The negation of the sentence above is

$$(x)\ (Gx \supset Hx) \cdot (x)\ (Fx \supset Gx) \cdot -(x)\ (Fx \supset Hx).$$

In other words, we would need to show that the sentence $-(x)\ (Fx \supset Hx)$ is inconsistent with the two premises.

$-(x)\ (Fx \supset Hx)$ is equivalent to $(\exists x)\ (Fx \cdot -Hx)$, so the problem is to prove that the set of three sentences

(1) $(x)\ (Gx \supset Hx)$
(2) $(x)\ (Fx \supset Gx)$
(3) $(\exists x)\ (Fx \cdot -Hx)$

is inconsistent. Sentence (3) states that there is at least one individual which is both F and not-H, so let us suppose an interpretation in which an individual 'a' is one of these. So we can write

(4) $Fa \cdot -Ha$

Sentences (1) and (2) contain statements which are true for all individuals in the interpretation, thus certainly including the designated individual 'a'. Thus we can deduce sentences (5) and (6):

(5) $Ga \supset Ha$

(6) $Fa \supset Ga$

In the propositional calculus, the expression $p \supset q \cdot q \supset r \cdot \supset \cdot p \supset r$ is logically true. (In future, I shall assume a knowledge of the theorems of propositional calculus.) So sentences (5) and (6) enable us to deduce (7):

(7) $Fa \supset Ha$

'Fa' is true, according to sentence (4) so we conclude

(8) Ha

But again sentence (4) gives us

(9) -Ha,

and we have arrived at a contradiction between sentences (8) and (9).

The working shows that the conjunction of (1), (2) and (3) is inconsistent, since in any interpretation which makes (1), (2) and (3) simultaneously true, there will be at least one individual like 'a'. The whole proof is conveniently summed up as follows:

A.	(1)	$(x)(Gx \supset Hx)$	
	(2)	$(x)(Fx \supset Gx)$	
	(3)	$(\exists x)(Fx \cdot -Hx)$	
	(4)	$Fa \cdot -Ha$	(3)
	(5)	$Ga \supset Ha$	(1)
	(6)	$Fa \supset Ga$	(2)
	(7)	$Fa \supset Ha$	(5), (6)
	(8)	Ha	(7), (4)
	(9)	-Ha	(4)

The numbers to the right of a line show from which lines it was inferred.

Here is an even simpler proof of the same sort:

B.	(1)	$(x)Fx$	
	(2)	$(\exists x)-Fx$	
	(3)	-Fa	(2)
	(4)	Fa	(1)

This shows that $(x)Fx \cdot (\exists x)\text{-}Fx$ is inconsistent, not surprisingly. That conjunction is a negation of

$$(x)Fx \supset \text{-}(\exists x)\text{-}Fx$$

or

$$(x)Fx \supset (x)Fx.$$

Now consider a conjunction $(\exists x)Fx \cdot (\exists x)\text{-}Fx$. This should not be inconsistent, of course. Some things may consistently be F while others are not-F. In that case, the following purported proof of inconsistency must be wrong:

C.	(1)	$(\exists x)Fx$	
	(2)	$(\exists x)\text{-}Fx$	
	(3)	Fa	(1)
	(4)	$\text{-}Fa$	(2)

The mistake lies in instantiating both lines (1) and (2) with the *same* individual 'a'-this is 'unfair' to the premises, which only assert that some are F and some are not-F, not that some are both F and not-F. The remedy for this is to instantiate line (2) with a different individual, say 'b':

D.	(1)	$(\exists x)Fx$	
	(2)	$(\exists x)\text{-}Fx$	
	(3)	Fa	(1)
	(4)	$\text{-}Fb$	(2).

The contradiction now disappears.

The examples above already illustrate almost the entire proof procedure. The procedure is one which demonstrates inconsistency, the contradictory of logical truth. The method consists in 'instantiating' quantified premises, using as few instances as possible, but observing always the rule that each new existential quantifier must be instantiated with a new instance. When the quantifiers have been removed, a contradiction is sought among the instantiated lines, which is a simple problem now in propositional calculus. If there is a contradiction, that means that the original premises are inconsistent. Clearly keeping the instances as few as possible increases the chances of finding a contradiction: compare C and D above.

I will now give further examples of the use of the method with a commentary to illustrate and justify it.

The first example shows that $(y)(\exists x)Fxy$ follows from $(\exists x)(y)Fxy$. The negation of $(y)(\exists x)Fxy$ is $(\exists y)(x)\text{-}Fxy$. So we need to show that the conjunction of premisses

E.	(1)	$(\exists x)(y)Fxy$
	(2)	$(\exists y)(x)\text{-}Fxy$

is inconsistent.

The existential quantifiers are instantiated differently, since they do not indicate that any individual satisfies both of them.

(3)	$(y)Fay$	(1)
(4)	$(x)\text{-}Fxb$	(2)

The universal quantifiers may be instantiated with 'a' or 'b' or both, since they state something true of *any* individuals. Here it is obvious that a contradiction will appear if (3) is instantiated with 'b' and (4) with 'a':

(5)	Fab	(3)
(6)	-Fab	(4)

I am justified in concluding now that (1) and (2) are inconsistent, since 'a' and 'b' could stand for *any* individuals in *any* interpretation which makes (1) and (2) both true. We have therefore shown that

$$(\exists x)\,(y)Fxy \supset (y)\,(\exists x)Fxy$$

is logically true.

Let us investigate the converse case

$$(y)\,(\exists x)Fxy \supset (\exists x)\,(y)Fxy.$$

The negation of the apodosis is $(x)\,(\exists y)\text{-}Fxy$, so we need to consider

F.	(1)	$(y)\,(\exists x)Fxy$
	(2)	$(x)\,(\exists y)\text{-}Fxy$

The universal quantifiers must be instantiated first, with the same individual for economy:

(3)	$(\exists x)Fxa$	(1)
(4)	$(\exists y)\text{-}Fay$	(2)

Now two more individuals are needed to instantiate the existential quantifiers in new ways:

(5)	Fba	(3)
(6)	-Fac	(4)

Obviously no contradiction could be found here, so we can only conclude that $(\exists x)\,(y)Fxy$ does not follow from $(y)\,(\exists x)Fxy$.

Now for a more complex example. We investigate the inconsistency or otherwise of the following premises:

$$(x)\,(\exists y)Fxy \supset (\exists x)\,(Gx \cdot Hx)$$
$$(x)\,(Hx \equiv Kx)$$
$$(x)\,(y)\,(Gx \supset Fxy)$$
$$(x)\,(y)\,(Gx \cdot \text{-}Ky)$$

A problem arises with the first premise, since in it there are quantifiers which do not have the whole sentence in their scope. Is it possible, for instance, to instantiate the first universal quantifier as 'a' and infer:

$$(\exists y)Fxy \supset (\exists x)\,(Gx \cdot Hx)\,?$$

It is not, as a counterexample shows. Suppose that $(\exists x)\,(Gx \cdot Hx)$ is false in some interpretation; then, since the premise is assumed true, $(x)(\exists y)Fxy$ must also be false. But $(\exists y)$-Fay might still be true, even in this case: this particular individual 'a' might make it true, though not *all* individuals do. So the premise might be true, but the instantiation false. It is not safe to instantiate quantifiers which do not have the whole line in their scope.

That particular premise may be written as an alternation, of course:

$$-(x)(\exists y)Fxy \vee (\exists x)(Gx \cdot Hx)$$

or

$$(\exists x)(y)\text{-}Fxy \vee (\exists x)(Gx \cdot Hx).$$

In a true alternation, either one or the other side is true, or both are. So each side may be conjoined separately with the other premises, and the inconsistency tested. If a contradiction arises in *both* cases, it must arise with the alternation as a whole. The complete working will make this clear:

G. (1) $(x)(Hx \equiv Kx)$

(2) $(x)(y)(Gx \supset Fxy)$

(3) $(x)(y)(Gx \cdot \text{-}Ky)$

(4) $(\exists x)(y)\text{-}Fxy \vee (\exists x)(Gx \cdot Hx)$

(101)	$(\exists x)(y)\text{-}Fxy$		(201)	$(\exists x)(Gx \cdot Hx)$	
(102)	$(y)\text{-}Fay$	(101)	(202)	$Gb \cdot Hb$	(201)
(103)	$\text{-}Faa$	(102)	(203)	$Hb \equiv Kb$	(1)
(104)	$Ga \supset Faa$	(2)	(204)	$Gb \cdot \text{-}Kb$	(3)
(105)	$Ga \cdot \text{-}Ka$	(3)	(205)	Kb	(202), (203)
(106)	Ga	(105)	(206)	$\text{-}Kb$	(202), (204)
(107)	Faa	(104)			
	contradicts	(103)			

It will be seen how one side of the alternation is tested in each branch. The numbering system is pure convenience. A contradiction arises in both branches, so the premises (1)–(4) are inconsistent. Notice how splitting up the alternation means that only quantifiers with whole lines in their scope are ever instantiated. Since any connective can be expressed in terms of negation, conjunction, and alternation, this 'branching' method can deal with any problem involving scope.

Axioms and Identity

It is of course possible to give a syntactic justification of this method, as Quine does in Chapter 29 of his (1974). The method itself can combine easily with an axiomatic system, to provide extensions of the lower predicate calculus (LPC). As an illustration, I will show how the method may be extended to deal with LPC plus identity.

The following axioms are sufficient for identity:

(1) $(x)(x=x)$

(2) $(x)(y)(\text{“}x\text{”}\cdot x=y\cdot\supset\text{“}y\text{”})$

The second axiom expresses 'Leibniz' Law'. Here "x" stands for some sentence containing x, while "y" is the same sentence with y in place of x. (I am assuming, of course, restrictions to prevent unwanted binding of variables.) The second axiom is in fact a *schema*: in any particular proof, some suitable sentence replaces "x".

Now these axioms may be added as *premises* where necessary in the course of a proof involving identity. I will give an illustration of this. The usual logical form given for "there is one and only one x such that Fx" is $(\exists x)(y)(Fy\equiv\cdot x=y)$. Allowod, Andersson and Dahl (1977) give the version, $(\exists x)(y)(Fx : Fy\supset\cdot x=y)$. Are these equivalent? In other words, we wish to see if the equivalence

$$(\exists x)(y)(Fy\equiv\cdot x=y)\equiv(\exists x)(y)(Fx : Fy\supset\cdot x=y)$$

is logically true.

Since $-(p\equiv q)$ is equivalent to $p.-q.v.-p.q$, there is an alternation right at the start, necessitating two separate proofs. First, let us test

H. (1) $(\exists x)(y)(Fx : Fy\supset\cdot x=y)$

(2) $(x)(\exists y)(Fy\cdot x\neq y\cdot v\cdot -Fy\cdot x=y)$

(2) is the negation of $(\exists x)(y)(Fy\equiv\cdot x=y)$. It is of course best to begin by instantiating the existential quantifier, to economize instances as much as possible. The proof proceeds as follows:

(3)	$(y)(Fa : Fy\supset\cdot a=y)$	(1)
(4)	$(\exists y)(Fy\cdot a\neq y\cdot v\cdot -Fy\cdot a=y)$	(2)
(5)	$Fb\cdot a\neq b\cdot v\cdot -Fb\cdot a=b$	(4)
(6)	$Fa : Fb\supset\cdot a=b$	(3)
(7)	$Fb\cdot a\neq b\cdot v\cdot -Fb\cdot a=b$	(5)

(101)	$Fb\cdot a\neq b$		(201)	$-Fb\cdot .a=b$	
(102)	$Fb\supset\cdot a=b$	(6)	(202)	Fa	(6)
(103)	$a=b$	(101), (102)			
(104)	$a\neq b$	(101)			

The alternation is conveniently dealt with by branching in this case too. The left hand branch leads to a contradiction anyway, but not the right. An axiom may conveniently be introduced at this point, namely, the second axiom, with the schema instantiated like this:

$$(x)(y)(Fx\cdot x=y\cdot\supset Fy)$$

The right hand branch then continues as follows:

(203)	$(x)(y)(Fx\cdot x=y\cdot\supset Fy)$	Axiom
(204)	$Fa\cdot a=b\cdot\supset Fb$	(203)

(205)	Fb	(201), (202), (204)
(206)	$-Fb$	(201)

That provides the necessary contradiction.

The other proof needed to establish the equivalence is given below. Here Axiom 1 is needed.

I.	(1)	$(\exists x)(y)(Fy \equiv \cdot x=y)$	
	(2)	$(x)(\exists y)(-Fx \vee \cdot Fy \cdot x \neq y)$	
	(3)	$(y)(Fy \equiv \cdot a=y)$	(1)
	(4)	$(\exists y)(-Fa \vee \cdot Fy \cdot a \neq y)$	(2)
	(5)	$-Fa \vee \cdot Fb \cdot a \neq b$	(4)
	(6)	$Fb \equiv \cdot a=b$	(3)
	(7)	$-Fa \vee \cdot Fb \cdot a \neq b$	(5)

(101)	$Fb \cdot a \neq b$		(201)	$-Fa$	
(102)	Fb	(101)	(202)	$Fa \equiv \cdot a=a$	(3)
(103)	$a=b$	(6)	(203)	$(x)(x=x)$	Axiom
(104)	$a \neq b$	(101)	(204)	$a=a$	(203)
			(205)	Fa	(202), (204)
				contradicts	(201)

So the two versions of the formula are equivalent. It is sometimes convenient to save axioms until they are needed, as here, instead of including them among the other premises. This enables one to see precisely when and why they are needed.

Since identity axioms can be introduced in this way, it is clear that axioms for modal logic and set theory could be similarly dealt with. However, I will leave this to a future paper in which the general method will be applied to the full apparatus of intensional logic.

The Boolean Problem

If I stopped at this point, I might be accused of not having solved the Boolean problem. I have provided a mechanical test for validity, but no method for determining just what follows from a collection of logical premises, which the Boolean problem requires. However, there can be no decision procedure for this general problem, as far as I know. What the method does for you is to give a mechanical test for your *guesses* of what follows from a set of premises. That is a great deal, and as much as you could expect, I think, especially in such fields as intensional logic.

The method can often give important clues about "missing" premises, though. I give a simple example below.

J.	(1)	$(x)(Gx \supset Hx)$
	(2)	$(x)(Fx \supset Gx)$

(3) $(x)(Fx \supset -Hx)$

(4) $Ga \supset Ha$ (1)

(5) $Fa \supset Ga$ (2)

(6) $Pa \supset -Ha$ (3)

(7) $Fa \supset Ha$ (4), (5)

There is no contradiction here as it stands, but clearly the addition of a line 'Fa' would cause one. The weakest premise 'Fa' could come from would be '$(\exists x)Fx$', which would of course be instantiated first.

So the method can guide guesses as well as test them. There will be more examples of this in the sequel to this paper.

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